Math 3235 Probability Theory

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$$X_{t+1} = \Delta_t X_t$$

$$\Downarrow$$

$$X_{t+1} = \prod_{i=0}^{t} \Delta_i \quad X_0 =$$

$$X_{t+1} = e^{\sum_{i=0}^{T} \ln \Delta_i} X_0$$

$$Z_i = \ln \Delta_i \implies Z_i \in N(\mu, \sigma^2)$$

$$T_t = \sum_{i=0}^{t} \Delta_i = N(t\mu, t\sigma^2)$$

$$X_{t+1} = e^{T_t}$$

$$e^{-(\ln \delta - \mu)^2}$$

$$\mathbb{P}(N_t \leq n) = \mathbb{P}(T_n \geq t)$$

$$f_{T_n}(t) = \frac{t^{n-1}}{(n-1)!} e^{-t}$$

$$\int_t^{\infty} \frac{s^{n-1}}{(n-1)!} e^{-s} ds$$

$$y = s - t \qquad s = t + y$$

$$\int_0^{\infty} \frac{(t+y)^{n-1}}{(n-1)!} e^{-(t+y)} dy =$$

$$e^{-t} \int_0^{\infty} \frac{1}{(n-1)!} \sum_{i=0}^{n-1} t^i y^{n-i-1} \binom{n-1}{i} e^{-y} dy =$$

$$e^{-t} \sum_{i=0}^{n-1} \frac{1}{(n-1)!} \frac{(n-1)!}{i!\,(n-i-1)!} t^i \int y^{n-i-1} e^{-y} dy$$

X_i i.i.d. $\qquad \mathbb{E}(X_i) = \mu$

$$\overline{X}_N = \frac{1}{N} \sum_{i=1}^{N} X_i$$

$\overline{X}_N \to \mu$ is square mean

$$\mathbb{E}\left((\overline{X}_N - \mu)^2\right) \xrightarrow[N\to\infty]{} 0$$

$$f_{\overline{X}_N}(x)$$

$$\int (x-\mu)^2 f_{\overline{X}_N}(x)\, dx \xrightarrow[N\to\infty]{} 0$$

$$\mathbb{P}\left(|\overline{X}_N - \mu| \geq t \right) \leq \frac{Var(X_i)}{N t^2}$$

Definition

X_n are r.v. and Y is a r.v.

We say that

$$X_n \xrightarrow{P} Y$$

if $\forall \delta$

$$\lim_{n \to \infty} \mathbb{P}\left(|X_n - Y| > \delta \right) = 0$$

X_n converge in probability to Y.

In particular if

X_i are i.i.d. r.v.

$$\overline{X}_N = \frac{1}{N} \sum_{i=1}^{N} X_i$$

$$\mathbb{P}\left(\left| \overline{X}_N - \mu \right| > t \right) \leq \frac{Var(X_i)}{N t^2}$$

Law of Large Number:

if X_i are i.i.d

$$\bar{X}_N \xrightarrow{P} \mu$$

X_i are i.i.d., is it necessary?

$$Var(\bar{X}_N) = \frac{Var(X_i)}{N}$$

$$\mathbb{P}(|X - \mu| \geq t) \leq \frac{Var(X)}{t^2}$$

$$\mu = \mathbb{E}(X).$$

$$Y_i = X_i - \mu$$

$$Var(\bar{X}) = \frac{1}{N^2} \mathbb{E}\left(\left(\sum_{i=1}^{N} Y_i\right)^2\right) =$$

$$= \frac{1}{N^2} \sum_{i,j} \mathbb{E}(Y_i Y_j)$$

$$\mathbb{E}(Y_i Y_j) = 0 \qquad i \neq j.$$

$$\sum_{ij} \mathbb{E}(Y_i Y_j) = 2 \sum_{j>i} \mathbb{E}(Y_i Y_j) + \frac{\sum_i \mathbb{E}(Y_i^2)}{N \sigma^2}$$

$$\mathbb{E}(Y_i Y_j) = c e^{-\nu |j-i|}$$

$$2 \sum_{j>i} \mathbb{E}(Y_i Y_j) \leq 2 \sum_{i=1}^{N} \sum_{j=i}^{N} c e^{-\nu |j-i|}$$

$$K = \sum_{j=0}^{\infty} c e^{-\nu j}$$

$$\leq 2 N K$$

$$\mathrm{Var}(\bar{X}_N) \leq \frac{K_1}{N}$$

o

Back To Chebychev:

$$\mathbb{P}\left(|\bar{X}_N - \mu| \geq t\right) \leq \frac{\sigma^2}{N t^2}$$

$N \approx \frac{1}{N}$ $$\mathbb{P}\left(|\bar{X}_N - \mu| \geq \frac{\tau}{N}\right) \leq N \sigma^2$$

τ $$\mathbb{P}\left(|\bar{X}_N - \mu| \geq t\right) \leq \frac{\sigma^2}{N t^2}$$

$\tau = \sqrt{2} \frac{\sigma}{\sqrt{N}}$

$$\mathbb{P}\left(|\bar{X}_N - \mu| \geq t\right) \leq \frac{1}{2}$$

$$\sqrt{N}\left(\bar{X}_N - \mu\right) \to N(0, \sigma^2)$$

$$\frac{\sqrt{N}}{\sigma}\left(\bar{X} - \mu\right) \to N(0, 1)$$

$$\frac{\sqrt{N}}{\sigma}\left(\overline{X} - \mu\right) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{X_i - \mu}{\sigma}$$

$$Z_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{X_i - \mu}{\sigma}$$

$$Z_N \Longrightarrow \mathcal{N}(0, 1)$$